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MODELING AND ANALYSIS OF LINEAR SYSTEMS WITH MULTIPLICATIVE POI--ETC(U)

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MULTIPLICATIVE POISSON WHITE NOISE

Steven I. Marcus
Department of Electrical Engineering
The University of Texas at Austin
Austin, Texas 78712

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Steven I. Marcus
Department of Electrical Engineering
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ABSTRACT

Poisson-driven bilinear systems (or linear systems with multiplicative Poisson impulse noise) are considered. The Poisson-driven canonical extension is derived by means of the product integral; its properties and relationship to modeling questions are also discussed. Equations for the moments of the state are derived, and the resulting criteria for stochastic stability are presented.

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I. INTRODUCTION

Linear systems with Gaussian white multiplicative noise arise frequently as reasonable models in applications, and they have consequently been studied extensively in the literature [1-10, 16-19]. In some applications, however, systems are subject to impulsive disturbances which arrive randomly in time; these disturbances can be modeled as Poisson impulse (white) noise [11]. Fortunately, Poisson-driven Markov processes have been studied in depth, and a theory analogous to that for vector Ito stochastic differential equations driven by Brownian motion has been obtained [12-14]. However, there are some important differences between the two theories, and some of these have important implications for the work presented here.

In this paper, linear systems with multiplicative Poisson impulse noise (also called Poisson-driven bilinear systems) are considered. Using the notation of Snyder [13], a Poisson-driven bilinear system satisfies the integral equation

$$x(t) = x(t_o) + \int_{t_o}^{t} A(\sigma)x(\sigma)d\sigma + \int_{t_o}^{t}\int_{\mathcal{U}} b(\sigma, x(\sigma), U) \cdot M(d\sigma, dU) \tag{1}$$

or the differential equation

$$dx(t) = A(t)x(t)dt + \int_{\mathcal{U}} b(t, x(t), U)M(dt, dU) \tag{2}$$

where $x(t)$ is a random n-vector or an $n \times n$ matrix, $A(t)$ is a non-random $n \times n$ matrix, b is an n-vector or $n \times n$ matrix which is linear in x for fixed U, M is a time-space Poisson process with intensity $\lambda(t)$ [13, p. 145], and $\mathcal{U}$ is the mark space of the compound Poisson process associated with M. The last integral in (1) is the counting integral which has an evaluation given by

$$\int_{t_o}^{t}\int_{\mathcal{U}} b(\sigma, x(\sigma), U)M(d\sigma, dU) = \begin{cases} 0 & N(t) = 0 \\ \sum_{n=1}^{N(t)} b(\tau_n, x(\tau_n), U_n), & N(t) \geq 1 \end{cases} \tag{3}$$

where $N(t) = \int_{t_o}^{t}\int_{\mathcal{U}} M(d\sigma, dU)$ is the number of incident points during $[t_o, t)$

regardless of their mark and τ_n and U_n are the time of occurrence and mark of the n^{th} point (notice that $N(t)$ is assumed to be almost surely left-continuous). Thus x will have discontinuities at the times τ_n and the size of the discontinuity at τ_n is $b(\tau_n, x(\tau_n), U_n)$ (for further details, see [13, Section 4.2]). A crucial result for such systems is the following analog of Ito's differential rule (a special case of [13, p. 199]).

Lemma 1 (Differential Rule): Let $x(t)$ be a random n-vector valued process satisfying (1). Assume that the conditions [13, Section 4.2] for the existence of the counting integral hold, and let $\varphi(t,x)$ be bounded for t and x finite and have continuous first derivatives with respect to t and the components of x. Then, with probability one, the random process $\varphi(t,x(t))$ satisfies

$$\varphi(t,x(t)) = \varphi(t_o,x(t_o)) + \int_{t_o}^{t} \left[\frac{\partial\varphi(\sigma,x(\sigma))}{\partial\sigma} + \left\langle A(\sigma)x(\sigma), \frac{\partial\varphi(\sigma,x(\sigma))}{\partial x(\sigma)} \right\rangle \right] d\sigma$$

$$+ \int_{t_o}^{t} \int_{\mathcal{U}} [\varphi(\sigma,x(\sigma) + b(\sigma,x(\sigma),U)) - \varphi(\sigma,x(\sigma))] M(d\sigma,dU) \qquad (4)$$

where $\frac{\partial\varphi(\sigma,x)}{\partial x}$ denotes the gradient of $\varphi(\sigma,x)$ with respect to x, and $\langle\cdot,\cdot\rangle$ denotes the usual inner product of n-vectors.

In the remainder of the paper, this and other results will be used to study Poisson-driven bilinear systems. For simplicity of notation we will concentrate on the case in which the mark space $\mathcal{U} = \{U_1, \ldots, U_k\}$ is finite so that

$$\int_{\mathcal{U}} b(t,x(t),U) M(dt,dU) = \sum_{i=1}^{k} b(t,x(t),U_i) dN_i(t)$$

where $N_i(t)$ are independent Poisson processes which count the jumps of the U_i. However, similar results can be obtained for more general mark spaces. First, systems evolving on Lie groups and homogeneous spaces will be discussed with respect to some general modeling questions, and the Poisson-driven canonical extension will be studied. Then equations for the moments of the state of the system will be derived, and these will provide criteria for stochastic stability.

II. THE POISSON-DRIVEN CANONICAL EXTENSION

Consider a rotating rigid body whose orientation is described by the 3 x 3 direction cosine matrix X satisfying

$$\dot{X}(t) = \left\{\sum_{i=1}^{3} \left[f_i(t) + u_i(t)\right] A_i\right\} X(t) \tag{5}$$

where f_i are the components of the nominal angular velocity, A_i form a basis for the Lie algebra so (3) of 3 x 3 skew-symmetric matrices $(A_i + A_i' = 0)$, and u_i are continuous scalar controls [3,10]. Because $A_i' = -A_i$, it is obvious that $\frac{d}{dt}(X'(t)X(t)) = 0$ and $X'(t)X(t)$ is constant for all t. Hence if $X'(0)X(0) = I$ and $\det X(0) = 1$, then X(t) evolves on the Lie group $SO(3) = \{X \in \mathbb{R}^{3 \times 3} \mid X'X = I, \det X = +1\}$ for all t.

If the body is subject to torques due, for example, to random micrometeorite collisions, the u_i may be approximately modeled as Poisson impulse disturbances [11]. In this case, the calculus of Poisson-driven Markov processes must be incorporated; i.e., it is assumed that X satisfies an equation of the form (2) where the mark space $\mathcal{U}$ has three points U_1, U_2, U_3 corresponding to the inputs u_1, u_2, u_3. If it is assumed that $b(t,X,U_i) = B_i X$, where B_i is a 3 x 3 matrix, then x satisfies

$$dX(t) = A(t)\, dt + \sum_{i=1}^{3} B_i dN_i(t)\, X(t) \tag{6}$$

where $N_i(t)$ are independent Poisson processes which count the jumps of U_i and have intensities $\lambda_i(t)$. If (6) is to be the Poisson driven form of model (5), it is reasonable to require that the solution X(t) of (6) also evolve on the Lie group SO(3)--i.e., that $X'(t)X(t)$ be constant for all t. Conditions under which this is true can be deduced from the following theorem, the proof of which is an application of the differentiation rule (Lemma 1).

Theorem 1: Assume that X(t) satisfies (6). Then X(t) evolves on the Lie group $\{X : X'QX = \text{constant}\}$ with probability one if and only if

$$A'(t)Q + QA(t) = 0 \qquad \text{for all } t \tag{7}$$

and

$$B_i'Q + QB_i + B_i'QB_i = 0; \qquad i = 1,2,3. \tag{8}$$

Proof: The differentiation rule (Lemma 1) implies that

$$d(X'QX) = X'(A'Q + QA)Xdt + \sum_{i=1}^{3} \left[(X+B_iX)'Q(X+B_iX) - X'QX\right]dN_i$$

$$= X'(A'Q + QA)Xdt + \sum_{i=1}^{3} X'(B_i'Q + QB_i + B_i'QB_i)XdN_i \; .$$

Hence, $d(X'QX) = 0$ (and $X'QX$ remains constant) if and only if (7) and (8) hold.

Applying this result to the rigid-body orientation example in which $Q = I$, it is necessary that $A(t) = A'(t)$ and $B_i + B_i' + B_i'B_i = 0$. Notice that the skew-symmetric A_i matrices of (5) do not satisfy the latter condition. Thus the Poisson-driven form (6) which corresponds to (5) is not obtained from (5) by merely substituting $dN_i(t)$ for $u_i(t)dt$, so some other approach must consequently be used. This phenomenon is analogous to the widely discussed case in which the u_i are replaced with Gaussian white noise [7-9, 16-19].

One method for defining the solution of the deterministic equation (5) is via the product integral, which "injects" the functions $(f_i(t) + u_i(t))$ into the Lie group $SO(3)$ (see [15,16]). Consider the deterministic differential equation on [0 ,T]

$$\dot{X}(t) = \left[\sum_i f_i(t)A_i\right]X(t) \quad ; \quad X(0) = I \tag{9}$$

or

$$dX(t) = \left[\sum_i (da_i(t)) A_i\right] X(t) \quad ; \quad X(0) = I \tag{10}$$

where

$$a_i(t) = \int_0^t f_i(s)ds \ .$$

Here X and $\{A_i\}$ are n x n matrices, f_i are continuous scalar functions, and $\sum_i$ denotes $\sum_{i=1}^{k}$. Notice that in this case the a_i are Lipschitzian [17]. Let $\mathscr{L}$ be the Lie algebra generated by $\{A_1, \ldots, A_k\}$ and $G = \{\exp \mathscr{L}\}_G$ the corresponding connected matrix Lie group [4,7,8,10,20]. We define the mapping H_n from k-vector valued functions on [0,T] to G-valued functions on [0,T] by

$$(H_n(a))(t) = I \ , \quad (t = 0)$$

$$= \exp\left[\sum_i (a_i(t) - a_i(\ell 2^{-n}))A_i\right](H_n(a))(\ell 2^{-n}), \quad (t \geq 0, \ \ell = [2^n t]) \tag{11}$$

where [m] denotes the largest integer $\leq$ m. It is shown in [15,16] that $\lim_{n\to\infty} H_n$ exists uniformly on [0,T] and is equal to the transition matrix $\Phi_{\Sigma f_i A_i}$ which solves (9); i.e., the solution of (9) is

$$\Phi_{\Sigma f_i A_i}(t,0) = \lim_{n\to\infty} (H_n(a))(t)$$

$$= \lim_{n\to\infty} \exp[B(t, \ell 2^{-n})] \prod_{j=0}^{\ell-1} \exp[B((j+1)2^{-n}, j2^{-n})] \tag{12}$$

where

$$B(t_2, t_1) = \sum_i A_i \int_{t_1}^{t_2} f_i(s)ds \ . \tag{13}$$

McKean's approach [9] (subsequently extended by Lo [7,8]) to the Gaussian white noise problem involves the generalization of the product integral to the case in which

$$da_i(t) = f_i(t)dt + dw_i(t) \tag{14}$$

where w is a Brownian motion process with

$$E[w(t)w'(s)] = \int_0^{\min(t,s)} R(\tau)d\tau \ .$$

It is shown in [7-9] that $\lim_{n\to\infty} H_n$ converges uniformly on $[0,T]$ almost surely to the solution of the Ito equation

$$dX(t) = \left\{\left[\sum_i f_i(t)A_i + \frac{1}{2}\sum_{i,j} R_{ij}(t)A_iA_j\right]dt + \sum_i A_i dw_i(t)\right\}X(t) \tag{15}$$

$$X(0) = I$$

and that X evolves on G almost surely. Hence (15) can be considered the "Ito form" of (10) when a_i is given by (14) (there will be further justification for this point of view in the sequel).

We now consider a similar approach to the Poisson white noise problem. Thus we define, for $i = 1,\ldots,k$,

$$da_i(t) = f_i(t)dt + dN_i(t) \tag{16}$$

where N_i are independent Poisson processes defined as in (6) with respect to an underlying space-time Poisson process M (see (2)) whose mark space has k points, and f_i are continuous processes on $[0,T]$. Let

$$\begin{aligned} K(\Delta) &\triangleq \sum_i a_i(\Delta)A_i \\ &\triangleq \sum_i (a_i(t) - a_i(\ell 2^{-n}))A_i \\ &\triangleq \sum_i \left(\int_{\ell 2^{-n}}^{t} f_i(\sigma)d\sigma + N_i(\Delta)\right)A_i \end{aligned} \tag{17}$$

where

$$N_i(\Delta) = N_i(t) - N_i(\ell 2^{-n}) \ . \tag{18}$$

Also, define

$$X_n(t) \triangleq (H_n(a))(t) . \tag{19}$$

Then

$$X_n(t) - X_n(\ell 2^{-n}) = (\exp(K(\Delta)) - I)X_n(\ell 2^{-n}). \tag{20}$$

As $n \to \infty$, the length Δ of the interval $[\ell 2^{-n}, t]$ approaches zero. The following heuristic derivation relies on two facts:

(A1) Since the underlying space-time process M is uniformly orderly [13], n can be chosen large enough so that there is at most one jump in $[\ell 2^{-n}, t]$;

[A2] For n large enough, products of second order and higher in Δ and products of Δ and $N_j(\Delta)$ are negligible as compared with Δ and $N_j(\Delta)$.

Assumption (A1) implies that *all the* $N_i(\Delta)$ *but one (say* $N_j(\Delta)$*) are* zero, and

$$\begin{aligned} N_j(\Delta) &= 1, \text{ if there is a jump in } [\ell 2^{-n}, t] \\ &= 0, \text{ otherwise.} \end{aligned} \tag{21}$$

Hence, for n large, $K(\Delta)$ can be approximated by

$$K(\Delta) = N_j(\Delta)A_j + \sum_i f_i(t)\Delta . \tag{22}$$

Notice also that for $p = 2, 3, \ldots$

$$\begin{aligned} [N_j(\Delta)]^p &= 1, \text{ if there is a jump in } [\ell 2^{-n}, t] \\ &= 0, \text{ otherwise} \end{aligned} \tag{23}$$

so $[N_j(\Delta)]^p = N_j(\Delta)$. Using this fact and (A2), we obtain

$$\begin{aligned} e^{K(\Delta)} - I &= [I + (N_j(\Delta)A_j + \sum_i A_i f_i(t)\Delta) + \frac{1}{2!}(N_j(\Delta)A_j + \sum_i A_i f_i(t)\Delta)^2 + \ldots] - I \\ &\simeq \sum_i A_i f_i(t)\Delta + [I + N_j(\Delta)(A_j + \frac{1}{2!} A_j^2 + \ldots) - I] . \end{aligned} \tag{24}$$

The last term in (24) is $e^{A_j} - I$ if $N_j(\Delta) = 1$ and 0 if $N_j(\Delta) = 0$; i.e., it is equal to $(e^{A_j} - I)N_j(\Delta)$. Since the jump could have occurred in one of the other N_i processes instead (j was chosen arbitrarily), we have

$$e^{K(\Delta)} - I \cong \sum_i A_i f_i(t)\Delta + \sum_i (e^{A_i} - I)N_i(\Delta) \ . \tag{25}$$

Substituting this result into (20) leads us to conjecture the following theorem.

Theorem 2: The sequence X_n converges uniformly on $[0,T]$ almost surely to the G-valued stochastic process X which satisfies the Poisson-driven bilinear equation

$$dX(t) = \left[\sum_i A_i f_i(t)dt + \sum_i (e^{A_i} - I)dN_i(t)\right] X(t)$$

$$X(0) = I \tag{26}$$

Theorem 2 can be proved by making the preceding derivation rigorous, but our proof is much more direct.

Proof: If $N_i(t) = 0$, $i = 1,\ldots,k$, for all $t \in [0,T]$, then the theorem reduces to the deterministic result (9) - (13). Therefore, suppose that there is at least one jump in $[0,T]$, and choose n sufficiently large so that there is no more than one jump in any interval $[i2^{-n},(i+1)2^{-n}]$. This can be done for almost any sample path because the underlying space-time Poisson process is uniformly orderly and the total number of jumps in $[0,T]$, $\sum_i N_i(T) \triangleq N(T)$, is finite with probability one. Then

$$X_n(t) = \prod_{j=0}^{\ell} C_n^j(t) \tag{27}$$

where

$$C_n^j(t) = \exp[B((j+1)2^{-n},j2^{-n})] \quad , \text{ if there is no jump in } \Delta_j \tag{28}$$

$$C_n^j(t) = \exp[B((j+1)2^{-n},\tau_j)] \exp(A_{i_j}) \exp[B(\tau_j,j2^{-n})],$$

$$\text{if there is a jump in } N_{i_j} \text{ at time } \tau_j \in \Delta_j . \tag{29}$$

Here, $B(t_2, t_1)$ is defined in (13) and $\Delta_j = [j2^{-n}, (j+1)2^{-n}]$ (in the equations (27)-(29), $(\ell+1)2^{-n}$ is to be interpreted as t). Since the total number of jumps in $[0,T]$ is finite, there will only be jumps in a finite number of the intervals Δ_j. From the deterministic results (9)-(13), it follows that X_n converges uniformly on $[0,T]$ almost surely, and

$$\lim_{n\to\infty} X_n(t) = \Phi(t, \tau_{N(t)}) e^{A_{i_{N(t)}}} \left\{ \prod_{j=1}^{N(t)-1} \Phi(\tau_{j+1}, \tau_j) e^{A_{i_j}} \right\} \Phi(\tau_1, 0) \tag{30}$$

where $\Phi \overset{\Delta}{=} \Phi_{\Sigma f_i A_i}$ is the solution to (9), $N(t) = \sum_i N_i(t)$, and there is a jump in N_{i_j} at time τ_j (for $j = 1, \ldots, N(t)$). It can easily be verified that (30) is the solution of the equation

$$X(t) = I + \sum_i A_i \int_0^t f_i(s) X(s) ds + \sum_{j=1}^{N(t)} (e^{A_{i_j}} - I) X(\tau_j) \ . \tag{31}$$

However, (3) implies that (31) and (26) are equivalent, and the theorem follows.

In [17,18], McShane defines the "canonical extension" in order to build a consistent theory for stochastic differential equations driven by almost surely continuous noise processes, such as Brownian motion (but <u>not</u> including Poisson processes). For example, the Ito equation (15) is the canonical extension of the deterministic (or Lipschitzian noise) equation

$$\dot{X}(t) = \left\{ \sum_i [f_i(t) + u_i(t)] A_i \right\} X(t); \quad X(0) = I \tag{32}$$

if the noises are Brownian motion processes. McShane also shows that the canonical extension possesses many desirable properties. Accordingly, we call (26) the <u>Poisson-driven canonical extension</u> of (32) and show that it possesses some of the same properties.

Returning to the rigid-body orientation example, if $A_i = -A_i'$, then defining $\tilde{B}_i \overset{\Delta}{=} e^{A_i} - I$ implies that

$$\tilde{B}_i + \tilde{B}_i' + \tilde{B}_i'\tilde{B}_i = (e^{A_i} - I) + (e^{-A_i} - I) + (e^{-A_i} - I)(e^{A_i} - I) = 0 \ .$$

Thus the Poisson-driven canonical extension (26) corresponding to (5) does preserve the property of evolving on SO(3) (see Theorem 1). This fact can be generalized as follows (the proof follows easily from Theorem 1).

Corollary 1: Assume that

$$A_i'Q + QA_i = 0 \ , \quad i = 1,\ldots,k \ . \tag{33}$$

Then the solutions $X(t)$ of both (26) and (32) satisfy $X'(t)QX(t) = Q$.

Thus the canonical extension preserves the property of evolution on a Lie group (this result is easily extended to homogeneous spaces).

As noted by McShane [17], another desirable property is the preservation of the adjoint property. That is, if the adjoint of (32) is defined by

$$\dot{Y}(t) = \left\{\sum_i [f_i(t) + u_i(t)](-A_i')\right\} Y(t)$$

$$Y(0) = I \tag{34}$$

then $\frac{d}{dt} X'(t)Y(t) = 0$, and

$$X'(t)Y(t) = X'(0)Y(0) = I \ , \quad t \geq 0 \ . \tag{35}$$

That the canonical extension preserves this property is shown in the next theorem.

Theorem 3: Consider the canonical extension (26) of (32), and define the canonical extension of (34):

$$dY(t) = \left[\sum_i (-A_i')f_i(t)dt + \sum_i (e^{-A_i'} - I)dN_i(t)\right] Y(t) \ .$$

$$Y(0) = I \tag{36}$$

Then, with probability one, the solutions of (26) and (36) satisfy

$$X'(t)Y(t) = X'(0)Y(0) = I \ , \quad t \geq 0 \ . \tag{37}$$

Proof: By the differentiation rule (Lemma 1),

$$d(X'Y) = X'\left(\sum_i (-A_i')f_i dt\right)Y + X'\left(\sum_i A_i' f_i dt\right)Y$$

$$+ \sum_i \left[(X + (e^{A_i} - I)X)'(Y + (e^{-A_i'} - I)Y) - X'Y\right]dN_i$$

$$= \sum_i \left[X'e^{A_i'}e^{-A_i'}Y - X'Y\right]dN_i = 0$$

Example 1: This example is analogous to one presented by McShane [17, p. 44], and it illustrates some other properties of the Poisson-driven canonical extension. Assume that the model, for scalar Lipschitzian disturbances a with $a(0) = 0$, is

$$x_1(t) = \int_0^t da(s) \tag{38}$$

$$x_2(t) = \int_0^t x_1(s)\, da(s)\ . \tag{39}$$

Putting this in the form (10) where x is a 3-vector ($x_3(t) = 1$), the model becomes

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \\ dx_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} da(t) \triangleq Ax(t)da(t) \tag{40}$$

$$x_1(0) = x_2(0) = 0 \ ; \ x_3(0) = 1$$

McShane shows that $\frac{1}{2}a^2(t)$ is the solution for x_2 in both (40) and, if a is a Brownian motion, of its Ito canonical extension.

Consider now the Poisson-driven canonical extension (26) of (40) (i.e., assume that a is a Poisson process):

$$dx(t) = (e^A - I)x(t)dN(t)$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} x(t)dN(t) \tag{41}$$

However, by the differential rule (Lemma 1), $z(t) = \frac{1}{2}(N^2(t))$ satisfies

$$dz(t) = \tfrac{1}{2}[(N(t) + 1)^2 - N^2(t)]dN(t)$$

$$= (N(t) + \tfrac{1}{2})dN(t) \tag{42}$$

which is the same as the x_2 equation in (41). Hence $\frac{1}{2}a^2(t)$ also solves the Poisson-driven canonical extension of (40) when a is a Poisson process. As noted by McShane, this solution has two important properties: consistency, since $\frac{1}{2}a^2(t)$ solves both (40) and its canonical extensions; and stability, since two noise processes that are, in some reasonable sense, "almost the same" will result in solutions for x_2 which are "almost the same."

Thus, this work can be viewed as an extension of a portion of McShane's results to Poisson-driven bilinear systems. A more detailed treatment of the relationship between this work and that of McShane will be presented in another paper, in which the extension of the Poisson-driven canonical extension to more general nonlinear systems will be presented.

III. MOMENT EQUATIONS AND STOCHASTIC STABILITY

In this section it is assumed, for simplicity, that the n-vector x satisfies the Poisson-driven bilinear system

$$dx(t) = Ax(t) + Bx(t)\, dN(t) + g\,dM(t) \tag{43}$$

where g is an n-vector, A and B are n x n matrices, and N and M are independent homogeneous Poisson processes with intensities λ_1 and λ_2, respectively. In order to investigate the moments of (43), we follow Refs. [3-5,10] in defining $x^{[p]}$ to be the vector of p^{th} order moments of x, and $A_{[p]}$ and $A^{[p]}$ to be the matrices which satisfy

$$\dot{x}(t) = Ax(t) \Rightarrow x^{[p]}(t) = A_{[p]}x^{[p]}(t) \tag{44}$$

and

$$y = Ax \Rightarrow y^{[p]} = A^{[p]}x^{[p]} \quad . \tag{45}$$

With this notation, the moment equations for $x(t)$ are derived by applying the differentiation rule (Lemma 1) to (43):

$$dx^{[p]}(t) = A_{[p]}x^{[p]}(t)dt + [(I + B)^{[p]} - I^{[p]}]x^{[p]}(t)dN(t)$$

$$+ [(x(t) + g)^{[p]} - x^{[p]}(t)]dM(t) \tag{46}$$

where I denotes the n x n identity matrix. Furthermore, from the properties of the counting integral [13, p. 196], it follows that

$$\frac{d}{dt} E[x^{[p]}(t)] = \{A_{[p]} + \lambda_1[(I + B)^{[p]} - I^{[p]}]\}E[x^{[p]}(t)]$$

$$+ \lambda_2 \; E[(x(t) + g)^{[p]}] - E[x^{[p]}(t)] \quad . \tag{47}$$

It can easily shown that the term $E[(x(t) + g)^{[p]}] - E[x^{[p]}(t)]$ is a linear combination of $E[x^{[i]}(t)]$, $i = 1,\ldots,p-1$ (notice that $E[x^{[p]}(t)]$ does not appear in this expression after it has been simplified). Hence

$$\frac{d}{dt}E\begin{bmatrix} x(t) \\ x^{[2]}(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x^{[p]}(t) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & & & & \\ \tilde{A}_{21} & \tilde{A}_{22} & & 0 & \\ \cdot & \cdot & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ \tilde{A}_{p1} & \tilde{A}_{p2} & \cdot\ \cdot & \cdot & \tilde{A}_{pp} \end{bmatrix} E\begin{bmatrix} x(t) \\ x^{[2]}(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x^{[p]}(t) \end{bmatrix} \tag{48}$$

where

$$\tilde{A}_{ii} = A_{[i]} + \lambda_1[(I + B)^{[i]} - I^{[i]}] \ . \tag{49}$$

This representation of the moment equations yields the following theorem, which is the direct analog of the Gaussian white noise result of Brockett [5, Theorem 4].

<u>Theorem 4:</u> Let the process $x(t)$ satisfy (43), and assume that $E[c(x(0))]^p$ exists for all linear functionals c and for $p = 1, 2, \ldots.$ Then

(i) $E[c(x(t))]^p$ exists for all $0 \le t < \infty$ and all linear functionals c;

(ii) there exist constants M_p and λ_p such that

$$E[c(x(t))]^p \le M_p e^{\lambda_p t}(1 + \|E[x(0)]\|^p) \ . \tag{50}$$

The representation (48) also shows that the stability of the moments of (43) depends only on the eigenvalues of the $\tilde{A}_{ii}$; however, no simple necessary and sufficient conditions for stability in terms of A and B are available. A sufficient condition appears in the following theorem.

<u>Theorem 5:</u> Let the process $x(t)$ satisfy (43), and assume that there exists a symmetric positive definite Q such that

$$B'QB + B'Q + QB = 0 \ ; \ A'Q + QA < 0 \ . \tag{51}$$

Then the moment equations are asymptotically stable--i.e., all moments $x^{[p]}(t)$ of (43) which exist initially approach a constant value as t approaches infinity, and this value is independent of the initial distribution.

<u>Proof:</u> First notice that the stability of (48) is not altered if we set $g = 0$; so we assume this condition holds. Also, the stability of the moments of

$$y(t) = Q^{\frac{1}{2}} x(t) \tag{52}$$

and $x(t)$ are the same. If x satisfies (43) and (51), then it is straightforward to show that y satisfies (43) and (51) with $Q = I$. Then Lemma 1 and (46) imply

$$d[y^{[p]'}y^{[p]}] = y^{[p]'}[A'_{[p]} + A_{[p]}]\, y^{[p]}dt$$

$$+ \Big\{y^{[p]'}[I^{[p]} + (I + B)^{[p]} - I^{[p]}]'\,[I^{[p]} + (I + B)^{[p]} - I^{[p]}]\, y^{[p]}$$

$$- y^{[p]'}y^{[p]}\Big\}dN(t)$$

$$= y^{[p]'}[A'_{[p]} + A_{[p]}]\, y^{[p]}dt$$

$$+ \Big\{y^{[p]'}(I + B' + B + B'B)^{[p]}\, y^{[p]} - y^{[p]'}\, y^{[p]}\Big\}dN(t)$$

$$= y^{[p]'}[A'_{[p]} + A_{[p]}]\, y^{[p]}dt \tag{53}$$

Hence,

$$\frac{d}{dt}E[y^{[p]'}(t)y^{[p]}(t)] = E[y^{[p]'}(t)(A'_{[p]} + A_{[p]})\, y^{[p]}(t)] \le 0 \tag{54}$$

and the right-hand side of (54) equals zero if and only if $y^{[p]}(t) = 0$ almost surely, which is true if and only if $E[y^{[p]'}(t)y^{[p]}(t)] = 0$. A Lyapunov-type stability argument implies that $E[y^{[p]'}(t)y^{[p]}(t)] \to 0$ as $t \to \infty$; hence the same is true for $E[y^{[p]}(t)]$, and the theorem is proved.

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Poisson-driven bilinear systems (or linear systems with multiplicative Poisson impulse noise) are considered. The Poisson-driven canonical extension is derived by means of the product integral; its properties and relationship to modeling questions are also discussed. Equations for the moments of the state are derived, and the resulting criteria for stochastic stability are presented.

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